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ABSTRACT

The problem of the wave field radiated by a point source above a plane impedance boundary is discussed. Some limitations on Ingard's solution of this problem are pointed out.

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Two theoretical papers have appeared recently which deal with the problem of the wave field radiated by a point source above a plane impedance boundary.[1,2] The authors of these papers pointed out some deficiencies in Ingard's treatment of this problem;[3] however, not all of the questions arising from Ingard's investigation have as yet been answered. The purpose of this brief note is to point out some limitations on Ingard's solution which were not noted by the authors of the papers mentioned above, and also to clarify some of these authors' comments on Ingard's analysis.

The starting point of the present discussion is the approximate expression for F of Ingard[3] (whose notation, except where otherwise indicated, we adopt), obtained by neglecting the term $\bar{t}^2$ in the denominator of his Eq. 10. For convenience we shall confine our remarks to the case in which $\gamma_0|\beta|<<1$. (The general case is somewhat more complicated and not of sufficient interest to warrant detailed discussion here.) This restriction allows us to simplify matters by writing $1+\beta\gamma_0 \simeq 1$ in the denominator of Eq. 10 after expanding and dropping the t^2 term. The resulting formula for F can be written

$$1 - F = (\gamma_0+\beta)kr_2 \int_0^\infty e^{-kr_2 t} \, [(\gamma_0+\beta)^2+2it]^{-\frac{1}{2}}dt. \qquad (1)$$

An expression for F in terms of the complementary error function can now be obtained by carrying out the integration in Eq. 1.[4] The result is

$$F = 1 - \pi^{\frac{1}{2}}\sigma e^{\sigma^2} \operatorname{erfc}(\sigma) \quad , \qquad (2)$$

where $\sigma = (kr_2/2i)^{\frac{1}{2}} (\gamma_0+\beta)$. This is essentially Ingard's result (see Eq. 13 of Ref. 3) with $\sigma \equiv \rho^{\frac{1}{2}}$ and with the sign error in Ingard's definition of ρ (his Eq. 14) corrected.

We have partially reconstructed Ingard's analysis in order to illustrate the main point of this note, which is that the derivation of Eq. 2 is valid only when

$$- \pi/2 < \arg(\sigma) \leq \pi/4 , \tag{3}$$

or, what is equivalent, when

$$- \pi/4 < \arg(\gamma_0+\beta) \leq \pi/2 . \tag{4}$$

This is because when $\arg(\gamma_0+\beta) \leq - \pi/4$ the integration path in Eq. 1 crosses the branch line of the integrand; i.e., there exists a $t_0 \geq 0$ such that $(\gamma_0+\beta)^2 + 2it_0 \leq 0$. When this occurs the analysis leading to Eq. 2 breaks down.

In the light of the above remarks the discrepancy between Ingard's approximate solution for large $|\sigma|$ and the corresponding results of Refs. 2 and 3 is easily resolved. This discrepancy, which arises from the absence of the surface-wave term in Ingard's expression for the wave field, was ascribed by Chien and Soroka[2] to Ingard's use (in Eq. 2) of an asymptotic expansion of $\mathrm{erfc}(\sigma)$ which is valid only when $|\arg(\sigma)| < \pi/2$, instead of the complete expansion, including the surface-wave term, which is valid for all values of $\arg(\sigma)$. This explanation, however, is misleading; the expansion used by Ingard is, in view of the restriction given by Eq. 3, entirely appropriate to the range of validity of his solution. The true reason for the discrepancy is now apparent: it is simply that Ingard's analysis is not valid for that range of σ, namely the range $\arg(\sigma) \leq -\pi/2$ (or $\arg(\gamma_0+\beta) \leq -\pi/4$), for which the surface-wave term appears in the asymptotic expansion of the wave field for large $|\sigma|$.

Another question regarding Ingard's solution was raised by the present writer with regard to its applicability in the case of small $|\sigma|$ (see Ref. 1, p. 961, third paragraph). It is now clear that these comments (which were also questioned by Chien and Soroka) are inaccurate and should be disregarded. Instead, it is to be noted that the case of small $|\sigma|$ is included in the general condition given by Eq. 3 for the validity of Ingard's analysis.

Although the expression for F given by Eq. 2 is, strictly speaking, limited to the range of σ defined by Eq. 3, it can obviously be extended analytically to the range $-3\pi/4 \leq \arg(\sigma) \leq \pi/4$, which is equivalent to extending Ingard's approximate solution to the entire right-half of the complex β plane. The question then arises as to whether this is a valid extension; i.e., whether the extended function $F(\sigma)$ yields a valid approximation to the wave field for those values of σ which lie outside the range defined by Eq. 3. Although there is no obvious reason for believing that this is the case, Chien and Soroka have nevertheless shown that, when $\gamma_0 \ll 1$ (i.e., when the source and receiver are near the boundary) and $|\beta| \ll 1$, Ingard's results agree with those of Refs. 1 and 2, to which the restriction given by Eq. 3 does not apply.

It should be pointed out, however, that when $|\beta| \gtrsim 1$ such agreement is not necessarily obtained. Consider, for example, the case in which $\gamma_0 = 0$ (i.e., source and receiver on the boundary) and $\beta = -i|\beta|$, with $|\beta| \gtrsim 1$ and $|\beta|(kr)^{\frac{1}{2}} \gg 1$. The soft-boundary results of either Ref. 1

(Eqs. 20, 33 and 34) or Ref. 2 (Eqs. 1, 7, 21 and 23) show that in this case the dominant term in the asymptotic expansion of the wave field is the surface-wave term. The corresponding result from Ingard's theory is obtained with the aid of Eq. 2 and the asymptotic expansion of $\mathrm{erfc}(\sigma)$ (this result can also be obtained from Eq. 28 of Ref. 2). It is readily seen that, because of the difference in the form of the surface-wave term arising from the asymptotic expansion of the complementary error function compared to that obtained either from Eq. 20 of Ref. 1 or Eq. 7 of Ref. 2, the results of Ingard's theory do not agree in this case with those of Refs. 1 or 2. (Note that this is true even if the asymptotic form of the Hankel function is used in the surface-wave term given by either Ref. 1, Eq. 20, or Ref. 2, Eq. 7.) It is clear, therefore, that Ingard's solution, even if the proper form of the asymptotic expansion of the complementary error function is used in it, is not valid in this case.

The situation as regards the validity of Ingard's theory can be summarized as follows. Equation 2, which is essentially Ingard's approximate expression for F for the case in which $\gamma_0|\beta| << 1$ and $kr_2 >> 1$ (the latter condition arises from the neglect of the term t^2 in the demoninator of Ingard's Eq. 10), is valid provided that the condition given by Eq. 3 (or Eq. 4) holds. It has also been shown to be valid when $|\beta| << 1$, irrespective of the condition given by Eqs. 3 or 4, provided that $\gamma_0 << 1$. When $|\beta| \gtrsim 1$, however, the error in Ingard's solution may become considerable as $\arg(\beta) \to -\pi/2$.

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[3]U. Ingard, J. Acoust. Soc. Am. 23, 329-335 (1951).

[4]W. Gautschi, "Error Function and Fresnel Integrals" in Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, edited by M. Abramowitz and I. A. Stegun (Nation Bureau of Standards, 1974), Chap. 7, p. 302, Formula 7.4.8.